$$
\begin{aligned}
E_1: &\quad x_1 + x_2 \qquad\quad + 3x_4 = 4,\\
E_2: &\quad 2x_1 + x_2 - x_3 + x_4 = 1,\\
E_3: &\quad 3x_1 - x_2 - x_3 + 2x_4 = -3,\\
E_4: &\quad -x_1 + 2x_2 + 3x_3 - x_4 = 4,
\end{aligned}
$$

Gaussian elimination method can be applied on augmented matrices:

$$
\begin{array}{c}\tilde{A}^{(1)}\\ \left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4\\ 2 & 1 & -1 & 1 & 1\\ 3 & -1 & -1 & 2 & -3\\ -1 & 2 & 3 & -1 & 4 \end{array}\right]\end{array}
\Rightarrow
\begin{array}{c}\tilde{A}^{(2)}\\ \left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4\\ 0 & -1 & -1 & -5 & -7\\ 0 & -4 & -1 & -7 & -15\\ 0 & 3 & 3 & 2 & 8 \end{array}\right]\end{array}
$$

$$
\Rightarrow \left[\begin{array}{cccc:c} 1 & 1 & 0 & 3 & 4\\ 0 & -1 & -1 & -5 & -7\\ 0 & 0 & 3 & 13 & 13\\ 0 & 0 & 0 & -13 & -13 \end{array}\right]
$$

$\tilde{A}^{(k)}$ is the kth augmented matrix.

Example

Represent the linear system

$$
\begin{aligned}
E_1: &\quad x_1 - x_2 + 2x_3 - x_4 = -8, \\
E_2: &\quad 2x_1 - 2x_2 + 3x_3 - 3x_4 = -20, \\
E_3: &\quad x_1 + x_2 + x_3 = -2, \\
E_4: &\quad x_1 - x_2 + 4x_3 + 3x_4 = 4,
\end{aligned}
$$

as an augmented matrix and use Gaussian Elimination to find its solution.

Solution The augmented matrix is

$$
\tilde{A} = \tilde{A}^{(1)} = \left[\begin{array}{cccc:c}
1 & -1 & 2 & -1 & -8 \\
2 & -2 & 3 & -3 & -20 \\
1 & 1 & 1 & 0 & -2 \\
1 & -1 & 4 & 3 & 4
\end{array}\right]
$$

Performing the operations

$$(E_2 - 2E_1) \rightarrow (E_2),\ (E_3 - E_1) \rightarrow (E_3), \quad \text{and} \quad (E_4 - E_1) \rightarrow (E_4),$$

gives

$$\tilde{A}^{(2)} = \left[\begin{array}{rrrr:r} 1 & -1 & 2 & -1 & -8 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right]$$

The diagonal entry $a_{22}^{(2)}$, called the **pivot element**, is 0,

so the procedure cannot continue in its present form.

But operations $(E_i) \leftrightarrow (E_j)$ are permitted,

(The first row with non-zero second element (3th row for this example)

is substituted with the second row to obtain a new matrix.)

$$\tilde{A}^{(2)'} = \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 2 & 4 & 12 \end{array}\right]$$

Since x_2 is already eliminated from E_3 and E_4, $\tilde{A}^{(3)}$ will be $\tilde{A}^{(2)'}$

continue with the operation $(E_4 + 2E_3) \rightarrow (E_4)$,

$$\tilde{A}^{(4)} = \left[\begin{array}{cccc:c} 1 & -1 & 2 & -1 & -8 \\ 0 & 2 & -1 & 1 & 6 \\ 0 & 0 & -1 & -1 & -4 \\ 0 & 0 & 0 & 2 & 4 \end{array}\right]$$

$$x_4 = \frac{4}{2} = 2,$$

$$x_3 = \frac{[-4 - (-1)x_4]}{-1} = 2,$$

$$x_2 = \frac{[6 - x_4 - (-1)x_3]}{2} = 3,$$

$$x_1 = \frac{[-8 - (-1)x_4 - 2x_3 - (-1)x_2]}{1} = -7.$$

Illustration

This illustration show that when a system of equations has no unique solution. Consider two systems presented below:

$$\begin{aligned} x_1 + \ x_2 + \ x_3 &= 4, \\ 2x_1 + 2x_2 + \ x_3 &= 6, \\ x_1 + \ x_2 + 2x_3 &= 6, \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 + \ x_2 + \ x_3 &= 4, \\ 2x_1 + 2x_2 + \ x_3 &= 4, \\ x_1 + \ x_2 + 2x_3 &= 6. \end{aligned}$$

Corresponding augmented matrices are as:

$$\tilde{A} = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 2 & 2 & 1 & 6 \\ 1 & 1 & 2 & 6 \end{array}\right] \qquad \text{and} \qquad \tilde{A} = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 2 & 2 & 1 & 4 \\ 1 & 1 & 2 & 6 \end{array}\right]$$

Performing operations,

$(E_2 - 2E_1) \rightarrow (E_2)$ and $(E_3 - E_1) \rightarrow (E_3)$

results in:

$$\tilde{A} = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right] \quad \text{and} \quad \tilde{A} = \left[\begin{array}{ccc:c} 1 & 1 & 1 & 4 \\ 0 & 0 & -1 & -4 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

At this point, $a_{22} = a_{32} = 0$.

In this case, there is no suitable row for substitution with the second one.

The algorithm requires that the procedure be halted and no solution to either system is obtained.

$$\begin{aligned} x_1 + x_2 + \quad x_3 &= \quad 4, \\ -x_3 &= -2, \\ x_3 &= \quad 2, \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 + x_2 + \quad x_3 &= \quad 4, \\ -x_3 &= -4, \\ x_3 &= \quad 2. \end{aligned}$$

The first system has an infinite number of solutions and the second system has no solution.

Operation Counts

Multiplications/divisions

$$\text{total number of operations} = \frac{n^3}{3} + n^2 - \frac{n}{3}$$

n is the matrix dimension.

Additions/subtractions

$$\text{total number of operations} = \frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6}$$

For large n, the amount of computation increases with n in proportion to n^3

n	Multiplications/Divisions	Additions/Subtractions
3	17	11
10	430	375
50	44,150	42,875
100	343,300	338,250

HOMEWORK 5:

Exercise Set 6.1: 5

Pivoting Strategies

We found that a row interchange was needed when one of the pivot elements is $a_{kk}^{(k)}$ 0. To reduce round-off error, it is often necessary to perform row interchanges even when the pivot elements are not 0. This is described via the following example:

Example

Apply Gaussian elimination to the system

$$\begin{aligned} E_1 &: \quad 0.003000x_1 + 59.14x_2 = 59.17 \\ E_2 &: \quad 5.291x_1 - 6.130x_2 = 46.78, \end{aligned}$$

using four-digit arithmetic with rounding, and compare the results to the exact solution $x_1 = 10.00$ and $x_2 = 1.000$.

Solution The first pivot element, $a_{11}^{(1)} = 0.003000$, is small, and its associated multiplier,

$$m_{21} = \frac{5.291}{0.003000} = 1763.6\overline{6},$$

rounds to the large number 1764. Performing $(E_2 - m_{21}E_1) \to (E_2)$ and the appropriate rounding gives the system

$$\begin{aligned} 0.003000x_1 + 59.14x_2 &\approx 59.17 \\ -104300x_2 &\approx -104400, \end{aligned}$$

instead of the exact system, which is

$$\begin{aligned} 0.003000x_1 + 59.14x_2 &= 59.17 \\ -104309.37\overline{6}x_2 &= -104309.37\overline{6}. \end{aligned}$$

- The pivot element a_{11} is small in magnitude compared to a_{21} . So, the magnitude of multiplier m_{21} will be much larger than 1.
- Thus, round-off error in computation of a_{12} is multiplied by m_{21} when computing a_{22} . this leads to the enhancement of the original error.

round-off error has not yet been propagated. Backward substitution yields

$$x_2 \approx 1.001,$$

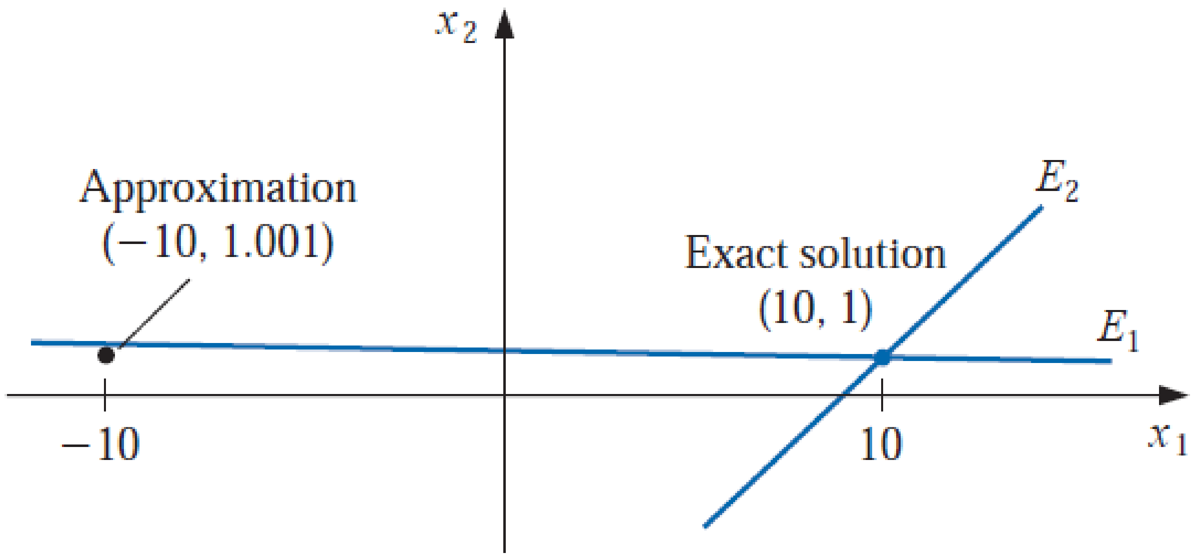
which is a close approximation to the actual value, $x_2 = 1.000$. However, because of the small pivot $a_{11} = 0.003000$,

$$x_1 \approx \frac{59.17 - (59.14)(1.001)}{0.003000} = -10.00$$

contains the small error of 0.001 multiplied by

$$\frac{59.14}{0.003000} \approx 20000.$$

This ruins the approximation to the actual value $x_1 = 10.00$.



Partial Pivoting

The simplest strategy is to select an element in the same column that is below the diagonal and has the largest absolute value. Then, the corresponding rows are interchanged.

Example

Apply Gaussian elimination to the system

$$\begin{aligned} E_1: &\quad 0.003000x_1 + 59.14x_2 = 59.17 \\ E_2: &\quad 5.291x_1 - 6.130x_2 = 46.78, \end{aligned}$$

using partial pivoting and four-digit arithmetic with rounding, and compare the results to the exact solution $x_1 = 10.00$ and $x_2 = 1.000$.

Solution The partial-pivoting procedure first requires finding

$$\max\left\{|a_{11}^{(1)}|, |a_{21}^{(1)}|\right\} = \max\{|0.003000|, |5.291|\} = |5.291| = |a_{21}^{(1)}|.$$

This requires that the operation $(E_2) \leftrightarrow (E_1)$ be performed to produce the equivalent system

$$\begin{aligned} E_1: &\quad 5.291x_1 - 6.130x_2 = 46.78, \\ E_2: &\quad 0.003000x_1 + 59.14x_2 = 59.17. \end{aligned}$$

The multiplier for this system is

$$m_{21} = \frac{a_{21}^{(1)}}{a_{11}^{(1)}} = 0.0005670,$$

and the operation $(E_2 - m_{21}E_1) \to (E_2)$ reduces the system to

$$5.291x_1 - 6.130x_2 \approx 46.78,$$
$$59.14x_2 \approx 59.14.$$

The four-digit answers resulting from the backward substitution are the correct values $x_1 = 10.00$ and $x_2 = 1.000$. ■

Illustration

The linear system

$$\begin{aligned} E_1: &\quad 30.00x_1 + 591400x_2 = 591700, \\ E_2: &\quad 5.291x_1 - \quad 6.130x_2 = 46.78, \end{aligned}$$

is the same as that previous examples except that all entries in the first equation have been multiplied by 1.E4. Solve this system using 4-digit arithmetic and partial pivoting.

The maximal value in the first column is 30.00, and the multiplier

$$m_{21} = \frac{5.291}{30.00} = 0.1764$$

leads to the system

$$\begin{aligned} 30.00x_1 + 591400x_2 &\approx 591700, \\ -104300x_2 &\approx -104400, \end{aligned}$$

which has the same inaccurate solutions as in Example 1: $x_2 \approx 1.001$ and $x_1 \approx -10.00$. $\square$

Scaled Partial Pivoting

This method places the element in the pivot position that is largest relative to the entries in its row.

The first step in this procedure is to define a scale factor s_i for each row as

$$s_i = \max_{1 \le j \le n} |a_{ij}|$$

Suppose that s_i is not zero (otherwise the system has no unique solution), pivoting for the first row is done by choosing p with,

$$\frac{|a_{p1}|}{s_p} = \max_{1 \le k \le n} \frac{|a_{k1}|}{s_k}$$

and performing $(E_1) \leftrightarrow (E_p)$.

Similar procedure is applied for other pivoting elements.

Illustration

For the system of equation (mentioned in the previous illustration):

$$\begin{aligned} E_1: \quad & 30.00x_1 + 591400x_2 = 591700, \\ E_2: \quad & 5.291x_1 - \quad 6.130x_2 = 46.78, \end{aligned}$$

Applying scaled partial pivoting gives

$$s_1 = \max\{|30.00|, |591400|\} = 591400$$

and

$$s_2 = \max\{|5.291|, |-6.130|\} = 6.130.$$

Consequently

$$\frac{|a_{11}|}{s_1} = \frac{30.00}{591400} = 0.5073 \times 10^{-4}, \qquad \frac{|a_{21}|}{s_2} = \frac{5.291}{6.130} = 0.8631,$$

and the interchange $(E_1) \leftrightarrow (E_2)$ is made.

Applying Gaussian elimination to the new system

$$5.291x_1 - 6.130x_2 = 46.78$$

$$30.00x_1 + 591400x_2 = 591700$$

produces the correct results: $x_1 = 10.00$ and $x_2 = 1.000$.

HOMEWORK 6:

Exercise Set 6.2: 13, 17 (parts c,d)